Soliton solutions for Q3

2008 J. Phys. A: Math. Theor. 41 142001

(http://iopscience.iop.org/1751-8121/41/14/142001)

IOP FTC ▶▶▶
IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **41** (2008) 142001 (11pp) doi:10.1088/1751-8113/41/14/142001

FAST TRACK COMMUNICATION

Soliton solutions for Q3

James Atkinson[1], **Jarmo Hietarinta**[2] **and Frank Nijhoff**[1]

[1] Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK
[2] Department of Physics, University of Turku, FIN-20014 Turku, Finland

Received 9 January 2008, in final form 8 February 2008
Published 26 March 2008
Online at stacks.iop.org/JPhysA/41/142001

Abstract
We construct *N*-soliton solutions to the equation called Q3 in the recent Adler–Bobenko–Suris classification. An essential ingredient in the construction is the relationship of $(\mathrm{Q3})_{\delta=0}$ to the equation proposed by Nijhoff, Quispel and Capel in 1983 (the NQC equation). This latter equation has two extra parameters, and depending on their sign choices we get a 4-to-1 relationship from NQC to $(\mathrm{Q3})_{\delta=0}$. This leads to a four-term background solution, and then to a 1-soliton solution using a Bäcklund transformation. Using the 1SS as a guide allows us to get the *N*-soliton solution in terms of the τ-function of the Hirota–Miwa equation.

PACS numbers: 02.30.Ik, 05.45.Yv

1. Introduction

Integrable lattice equations have a long history, going back to pioneering work in the 1970s [1, 2] with subsequent development of systematic approaches in the early 1980s [3–5] (cf also the review [6]). It has long been known, in fact it is implicit in these constructions, that the emerging examples possess the property of 'multidimensional consistency'. By this we mean the property that the partial difference equation describing the lattice systems can be consistently embedded in a multidimensional space, with in principle an infinity of lattice variables. Most succinctly this property was described in recent years in [7], in a context where it was explicitly used to achieve multidimensional reductions, and it was subsequently re-appraised in [8]. This multidimensional consistency is a very natural property, it is the precise analogue of the well-known existence of compatible higher time-flows in hierarchies of soliton systems, and as such it was quite well understood in the earlier publications mentioned.

In [9], the property of 'consistency around a cube' (CAC) was used to classify partial difference equations defined on an elementary square of a two-dimensional lattice. Remarkably, within certain conditions (symmetry and the 'tetrahedron condition'), a full list of scalar quadrilateral lattice equations could be established, and this list is surprisingly short. (In a more recent paper [10], the classification statement was proven under slightly less restrictive conditions.)

The list of CAC-integrable lattice equations in [9] contains some interesting examples, but the equation at the top of the list (denoted as Q4 in [9]) was actually found earlier by Adler (cf [11]). This equation, which we refer to as *Adler's equation*, is in fact an integrable discretization of the famous Krichever–Novikov (KN) equation [12, 13]. For Adler's equation a Lax pair was established in [14] on the basis of its multidimensional consistency, and in [15] this equation was shown to be a master equation among various well-known integrable systems associated with an elliptic curve. The first solutions for Adler's equation were established in a recent paper [16] (cf also [17] for a slight generalization of those results). However, so far little is known about the algebraic structure behind Adler's equation, and since this equation has lattice parameters which lie on an elliptic curve, the underlying structure is expected to be interesting but rather complicated. Thus, before tackling Adler's equation, it is of interest to study some of the other examples emerging from the list of [9], in order to see how the underlying structure of such equations can be revealed.

Here we focus on the equation denoted as Q3 in [9] and which is just below Q4 in the hierarchy. It is written as

$$\begin{aligned}\mathring{p}(1-\mathring{q}^2)(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}}) - \mathring{q}(1-\mathring{p}^2)(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}}) - (\mathring{p}^2-\mathring{q}^2)(\widehat{u}\widetilde{u}+u\widehat{\widetilde{u}}) \\ = \delta^2(\mathring{p}^2-\mathring{q}^2)(1-\mathring{p}^2)(1-\mathring{q}^2)/(4\mathring{p}\mathring{q}).\end{aligned} \tag{1.1}$$

In (1.1) we have adopted the convention of representing shifts in the rectangular lattice with tildes and hats. The corners of an elementary plaquette on a rectangular lattice are thus

$$u \equiv u_{n,m}, \qquad \widetilde{u} \equiv u_{n+1,m}, \qquad \widehat{u} \equiv u_{n,m+1}, \qquad \widehat{\widetilde{u}} \equiv u_{n+1,m+1}.$$

The *lattice parameters* $\mathring{p}, \mathring{q}$ in (1.1) are associated with the n, m directions in the lattice, respectively (they can be thought of as measuring the grid size in these directions), while δ is a global parameter.

In this paper we construct a general family of N-soliton solutions of Q3. The construction of these solutions is based on the relationship of (1.1) to a lattice equation that first appeared in [3] (cf also [4]) about 25 years ago, and which is equivalent to the $\delta = 0$ case of Q3. The explicit 4-to-1 relationship between these two equations is explained in section 2, and it requires the introduction of another parametrization which is more suitable for the solution. The N-soliton solution of Q3 with parameter δ (denoted by $(\mathrm{Q3})_\delta$) then appears as a *linear* combination of four independent solutions of the $\delta = 0$ case, $(\mathrm{Q3})_0$, with four arbitrary coefficients subject to one single relation. We have obtained this result in two different ways, one of which employs a novel Miura transformation (explained in the appendix), which allows us to derive soliton solutions for $(\mathrm{Q3})_\delta$ from the known solutions of $(\mathrm{Q3})_0$ of [3, 4]. We will not present this approach here, because it requires quite a bit of a notational machinery for which we lack space in this paper, and this approach will be published elsewhere [18].

Alternatively, we can use the 4-to-1 correspondence which appears through the introduction of some additional lattice shifts associated with the parametrization given in section 2, to obtain a general 'seed' solution for the relevant Bäcklund transformations. This allows us in section 3 to obtain the 1-soliton solution in a form which suggests a τ-function description. We will present the N-soliton solution of Q3 in Hirota form in section 4, and show how the solution is related to a set of discrete Hirota–Miwa equations in a four-dimensional lattice.

2. The basic lattice equations

The special case $\delta = 0$ of (1.1) appeared for the first time in [3] (cf also [4]) in the form

$$\frac{1+(p-a)s-(p+b)\widetilde{s}}{1+(q-a)s-(q+b)\widehat{s}} = \frac{1+(q-b)\widetilde{s}-(q+a)\widehat{\widetilde{s}}}{1+(p-b)\widehat{s}-(p+a)\widehat{\widetilde{s}}}. \tag{2.1}$$

We call this the NQC equation, following [19].

To bring equation (2.1) to the form of $(\mathrm{Q}3)_0$ we perform the transformation $(a+b \neq 0)$:

$$u_{n,m} = \tau^n \sigma^m \left(s_{n,m} - \frac{1}{a+b} \right), \tag{2.2}$$

where

$$\tau := \sqrt{\frac{(p+a)(p+b)}{(p-a)(p-b)}}, \qquad \sigma := \sqrt{\frac{(q+a)(q+b)}{(q-a)(q-b)}}. \tag{2.3}$$

This yields $(\mathrm{Q}3)_0$ with the parametrization

$$P(u\widehat{u} + \widetilde{u}\widehat{\widetilde{u}}) - Q(u\widetilde{u} + \widehat{u}\widehat{\widetilde{u}}) - (p^2 - q^2)(\widehat{u}\widetilde{u} + u\widehat{\widetilde{u}}) = 0, \tag{2.4}$$

where the lattice parameters have now become points $\mathfrak{p} = (p, P)$ and $\mathfrak{q} = (q, Q)$, respectively, on the (Jacobi) elliptic curve:

$$P^2 = (p^2 - a^2)(p^2 - b^2), \qquad Q^2 = (q^2 - a^2)(q^2 - b^2). \tag{2.5}$$

In this parametrization the $\delta \neq 0$ version of (1.1) reads

$$P(u\widehat{u} + \widetilde{u}\widehat{\widetilde{u}}) - Q(u\widetilde{u} + \widehat{u}\widehat{\widetilde{u}}) - (p^2 - q^2)(\widehat{u}\widetilde{u} + u\widehat{\widetilde{u}}) = \delta^2 \frac{(p^2 - q^2)}{4PQ}, \tag{2.6}$$

where the functions u in (1.1) and (2.6) are related by $u_{1.1} = (a^2 - b^2) u_{2.6}$.

Note that the original parametrization of Q3 in (1.1) is subtly different from (2.4) and they can be related by the identifications[1]:

$$\mathring{p}^2 = \frac{p^2 - b^2}{p^2 - a^2}, \qquad \mathring{q}^2 = \frac{q^2 - b^2}{q^2 - a^2}, \qquad P = \frac{(b^2 - a^2)\mathring{p}}{1 - \mathring{p}^2}, \qquad Q = \frac{(b^2 - a^2)\mathring{q}}{1 - \mathring{q}^2}. \tag{2.7}$$

It is important for later to observe that the parametrizations of $P, Q, \mathring{p}, \mathring{q}$ are invariant under the sign change of a and/or b while the NQC equation itself is not. This means that there are in fact *four different versions* of NQC (corresponding to these sign changes) and they all provide a different solution to $(\mathrm{Q}3)_0$, though the transformation (2.2) with corresponding sign changes in it and (2.3). We will use this 4-to-1 correspondence later to construct solutions to Q3.

It was shown in [3] that (2.1) interpolates through different choices of the auxiliary parameters a, b between various lattice equations 'of KdV type', and hence could be thought of as an interpolating equation. We can identify, e.g. the following subcases of (2.1) appearing in the list of [9] (up to gauge transformations, where necessary): $a = b = 0 \Rightarrow (\mathrm{Q}_1)_0$, $a = 0, b \to \infty \Rightarrow (\mathrm{H}_3)_0$ and $a, b \to \infty \Rightarrow \mathrm{H}_1$, which are respectively the lattice Schwarzian KdV equation, the lattice potential modified KdV equation and the lattice potential KdV equation. For (2.1) and all these subcases N-soliton solutions can be given in closed form, which follows as an immediate application of the direct linearization scheme elaborated in [3, 4].

[1] In fact, the two different parametrizations presented in (1.1) and (2.6) are just different ways to parametrize the equation $c_1(u\widehat{u} + \widetilde{u}\widehat{\widetilde{u}}) - c_2(u\widetilde{u} + \widehat{u}\widehat{\widetilde{u}}) - c_3(\widehat{u}\widetilde{u} + u\widehat{\widetilde{u}}) = c_4$, with the constraints arising from CAC (related to dependence of c_i on lattice parameters associated with different directions of the cube).

3. Background and one-soliton solutions

3.1. Seed or background solution for Q3

The trivial solution to (2.1) is $s_{n,m} \equiv 0$ and from this it follows that $u_{n,m} = c\tau^n\sigma^m$ is a background or 'seed' solution for $(\mathrm{Q3})_0$ for any constant c. Furthermore, as discussed before, by changing the signs of a and/or b we get other seed solutions for $(\mathrm{Q3})_0$, namely

$$u^{++}_{n,m} = A\tau^n\sigma^m, \qquad u^{--}_{n,m} = B\tau^{-n}\sigma^{-m}, \qquad u^{+-}_{n,m} = C\dot{\tau}^n\dot{\sigma}^m, \qquad u^{-+}_{n,m} = D\dot{\tau}^{-n}\dot{\sigma}^{-m}, \tag{3.1}$$

where τ and σ were defined in (2.3) and

$$\dot{\tau} := \sqrt{\frac{(p+a)(p-b)}{(p-a)(p+b)}}, \qquad \dot{\sigma} := \sqrt{\frac{(q+a)(q-b)}{(q-a)(q+b)}}. \tag{3.2}$$

Starting with one such seed solution for $(\mathrm{Q3})_0$ one can use a Miura transformation (see the appendix) to derive a solution for $(\mathrm{Q3})_\delta$. The result turns out to be a linear combination of three of the above terms, and leads us to try a linear combination of all four terms, that is

$$u_\theta \equiv u^{0SS}_{n,m} = A\tau^n\sigma^m + D\tau^{-n}\sigma^{-m} + B\dot{\tau}^n\dot{\sigma}^m + C\dot{\tau}^{-n}\dot{\sigma}^{-m}. \tag{3.3}$$

It is easy to verify that this is indeed a solution of $(\mathrm{Q3})_0$ provided that

$$AD(a+b)^2 - BC(a-b)^2 = 0, \tag{3.4}$$

and, perhaps surprisingly, that (3.3) is also a solution of $(\mathrm{Q3})_\delta$, provided that

$$AD(a+b)^2 - BC(a-b)^2 = -\frac{\delta^2}{16ab}. \tag{3.5}$$

From this last result we see that when $\delta \neq 0$ one cannot use any *single* seed given in (3.1) as a starting solution, and for a 'germinating seed' (in the terminology of [16]) one needs at least the pair $++$, $--$ or the pair $+-$, $-+$.

3.2. 1-Soliton solution $(\mathrm{Q3})_\delta$ *from BT*

Having obtained a nontrivial background solution (3.3), we can now proceed to construct soliton solutions starting from this seed solution of the Bäcklund transformation (BT). From cubic consistency it follows that we can impose in addition to the lattice equation (2.4) also the set of equations:

$$P(u\overline{u} + \widetilde{u}\widetilde{\overline{u}}) - K(u\widetilde{u} + \overline{u}\widetilde{\overline{u}}) = (p^2-k^2)\left(\widetilde{u}\overline{u} + u\widetilde{\overline{u}} + \frac{\delta^2}{4PK}\right), \tag{3.6a}$$

$$Q(u\overline{u} + \widehat{u}\widehat{\overline{u}}) - K(u\widehat{u} + \overline{u}\widehat{\overline{u}}) = (q^2-k^2)\left(\widehat{u}\overline{u} + u\widehat{\overline{u}} + \frac{\delta^2}{4QK}\right), \tag{3.6b}$$

where $K^2 = (k^2-a^2)(k^2-b^2)$, and the 'bar'-direction is for increasing number of solitons. We now search for a new $\overline{u}(\equiv u^{1SS})$ of the form: $\overline{u} = \overline{u}_\theta + v$, where $\overline{u}_\theta$ is the bar-shifted background solution (3.3)

$$\bar{u}_\theta = A\tau^n\sigma^m\kappa + D\tau^{-n}\sigma^{-m}\kappa^{-1} + B\dot{\tau}^n\dot{\sigma}^m\dot{\kappa} + C\dot{\tau}^{-n}\dot{\sigma}^{-m}\dot{\kappa}^{-1}, \tag{3.7}$$

where

$$\kappa = \sqrt{\frac{(k+a)(k+b)}{(k-a)(k-b)}}, \qquad \dot{\kappa} = \sqrt{\frac{(k+a)(k-b)}{(k-a)(k+b)}}. \tag{3.8}$$

(Note however that we can absorb the $\kappa, \dot{\kappa}$ factors into A, B, C, D without changing the relation (3.4) or (3.5).)

From (3.6) one can then solve

$$\widetilde{v} = \frac{[(p^2-k^2)\widetilde{u}_\theta + K\widetilde{\overline{u}}_\theta) - Pu_\theta]v}{-Kv + [P\widetilde{u}_\theta - K\overline{u}_\theta - (p^2-k^2)u_\theta]}, \qquad \widehat{v} = \frac{[(q^2-k^2)\widehat{u}_\theta + K\widehat{\overline{u}}_\theta) - Qu_\theta]v}{-Kv + [Q\widehat{u}_\theta - K\overline{u}_\theta - (q^2-k^2)u_\theta]} \tag{3.9}$$

where the expected δ^2 term in the numerator disappears by virtue of the definitions (3.7), (3.5). Then introducing $v = f/g$ we can linearize these equations and obtain

$$\widetilde{\phi} = \Lambda \begin{pmatrix} (p^2-k^2)\widetilde{u}_\theta + K\widetilde{\overline{u}}_\theta - Pu_\theta & 0 \\ -K & P\widetilde{u}_\theta - K\overline{u}_\theta - (p^2-k^2)u_\theta \end{pmatrix} \phi \tag{3.10}$$

and similarly for $\widehat{\phi}$, where $\phi = (f, g)^T$. The factor Λ is determined by the condition that the shifts with $\widetilde{\ }$ and $\widehat{\ }$ must commute, from which it follows that we should take $\Lambda \propto 1/U_\theta$, where (compare with (3.3)

$$(U_\theta)_{n,m} = (a+b)A\tau^n\sigma^m - (a+b)D\tau^{-n}\sigma^{-m} + (a-b)B\dot{\tau}^n\dot{\sigma}^m - (a-b)C\dot{\tau}^{-n}\dot{\sigma}^{-m}. \tag{3.11}$$

Then with some algebra we obtain

$$\phi_{n,m} = (p-k)^n(q-k)^m \begin{pmatrix} \rho_{n,m}(U_\theta)_{n,m}/(U_\theta)_{0,0} & 0 \\ \frac{K}{2k}[1-\rho_{n,m}]/(U_\theta)_{0,0} & 1 \end{pmatrix} \phi_{0,0}, \tag{3.12}$$

where $\rho_{n,m}$ is the discrete 'plane-wave factor', defined by

$$\rho_{n,m} = \left(\frac{p+k}{p-k}\right)^n \left(\frac{q+k}{q-k}\right)^m. \tag{3.13}$$

From result (3.12) we can reconstruct v of the 1SS:

$$v_{n,m} = \frac{2k(U_\theta)_{n,m}\rho_{n,m}v_{0,0}}{2k(U_\theta)_{0,0} + Kv_{0,0}(1-\rho_{n,m})}, \tag{3.14}$$

and finally $u^{1SS}_{n,m} = \bar{u}_\theta + v$.

For a more explicit form showing the A, B, C, D dependence, we incorporate a constant ρ_0 in (3.13) so that the denominator becomes proportional to $1 + \rho_{n,m}$. Furthermore, after scaling A, B, C, D so that $\bar{u}_\theta$ of (3.7) becomes u^{0SS} of (3.3) we can write the 1SS (3.14) as

$$\begin{aligned} u^{1SS}_{n,m} = \Bigg[& A\tau^n\sigma^m \left(1 + \rho_{n,m}\frac{(a-k)(b-k)}{(a+k)(b+k)}\right) + D\tau^{-n}\sigma^{-m}\left(1 + \rho_{n,m}\frac{(a+k)(b+k)}{(a-k)(b-k)}\right) \\ & + B\dot{\tau}^n\dot{\sigma}^m\left(1 + \rho_{n,m}\frac{(a-k)(b+k)}{(a+k)(b-k)}\right) \\ & + C\dot{\tau}^{-n}\dot{\sigma}^{-m}\left(1 + \rho_{n,m}\frac{(a+k)(b-k)}{(a-k)(b+k)}\right)\Bigg] \Big/ (1+\rho_{n,m}). \end{aligned} \tag{3.15}$$

4. *N*-soliton solutions and Hirota bilinear form

We will now present the main result of the paper, which is a general *N*-soliton solution of Q3. This solution can be written in the following form:

$$\begin{aligned} u_{n,m} = {} & A\tau^n\sigma^m \frac{F(n,m,\alpha+1,\beta+1)}{F(n,m,\alpha,\beta)} + D\tau^{-n}\sigma^{-m}\frac{F(n,m,\alpha-1,\beta-1)}{F(n,m,\alpha,\beta)} \\ & + B\dot{\tau}^n\dot{\sigma}^m\frac{F(n,m,\alpha+1,\beta-1)}{F(n,m,\alpha,\beta)} + C\dot{\tau}^{-n}\dot{\sigma}^{-m}\frac{F(n,m,\alpha-1,\beta+1)}{F(n,m,\alpha,\beta)}, \end{aligned} \tag{4.1}$$

where the τ-function F is given by

$$F(n, m, \alpha, \beta) = \sum_{\mu_j \in \{0,1\}} \exp\left(\sum_{j=1}^{N} \mu_j \eta_j + \sum_{1 \leqslant i < j \leqslant N} \mu_i \mu_j a_{ij}\right), \tag{4.2}$$

where

$$\exp \eta_j \equiv \rho_{nm\alpha\beta}(k_j) := \left(\frac{p + k_j}{p - k_j}\right)^n \left(\frac{q + k_j}{q - k_j}\right)^m \left(\frac{a - k_j}{a + k_j}\right)^\alpha \left(\frac{b - k_j}{b + k_j}\right)^\beta \rho_j^0, \tag{4.3}$$

$$\exp a_{ij} \equiv A_{ij} := \left(\frac{k_i - k_j}{k_i + k_j}\right)^2, \tag{4.4}$$

and the coefficients in (4.1) are restricted by

$$AD(a + b)^2 - BC(a - b)^2 = -\frac{\delta^2}{16ab}. \tag{4.5}$$

One way in which this result was derived is by using the Miura transformation given in the appendix and a number of relations following from the direct linearization structure of the NQC equation of [3, 4]. A full derivation following this line of argument is given in [17], and here, for lack of space, we will instead argue the result based on the connection with the Hirota–Miwa difference equations [20, 21].

Having established the 1SS (3.15), which is a special case of form (4.1) with (4.2), we checked the 2SS suggested by Hirota's perturbative approach

$$F(n, m, \alpha, \beta) = 1 + \rho_{nm\alpha\beta}(k_1) + \rho_{nm\alpha\beta}(k_2) + A_{12}\rho_{nm\alpha\beta}(k_1)\rho_{nm\alpha\beta}(k_2) \tag{4.6}$$

and verified that the phase factor A_{ij} is given as in (4.4). We have also explicitly verified that the 3SS version solves Q3.

The form of the phase factor (4.4) allows us to identify the solution to be in the Hirota–Miwa hierarchy [20, 21], within the reduction $-q_i = p_i \equiv k_i$ (up to some notational conventions for the parameters), except that we now seem to have a 4D system n, m, α, β with diagonal $[(\alpha \pm 1, \beta \pm 1)]$ and anti-diagonal $[(\alpha \pm 1, \beta \mp 1)]$ reductions.

The N-soliton F in (4.2) solves the Hirota–Miwa bilinear difference equations in any three of the four indices, e.g.

$$(q + a)F_1F_{23} - (a + p)F_2F_{13} + (p - q)F_{12}F_3 = 0, \tag{4.7a}$$

$$(q + b)F_1F_{24} - (b + p)F_2F_{14} + (p - q)F_{12}F_4 = 0, \tag{4.7b}$$

$$(p - q)F_{123}F + (q - a)F_{13}F_2 - (p - a)F_1F_{23} = 0, \tag{4.7c}$$

$$(p - q)F_{124}F + (q - b)F_{14}F_2 - (p - b)F_1F_{24} = 0. \tag{4.7d}$$

Here we have used the notation where only the shifted index is indicated, e.g. $F_{n,m+1,\alpha+1,\beta+1} \equiv F_{234}$. The shift in the negative direction is indicated by a bar, e.g. $F_{n,m,\alpha-1,\beta+1} \equiv F_{\bar{3}4}$. In the following we also need the various shifts of these equations. (The equations above would be more symmetric if we were to change $a \to -a$, $b \to -b$, cf (4.3).)

We now show that the expression (4.1) is a solution of equation (2.6), if F solves the HM equations (4.7). First, we note that since (2.6) is quadratic except for the inhomogenous term, the substitutions of (4.1) give terms quadratic in the coefficients A, B, C, D. Thus we have three types of terms: (i) squared terms that go with A^2, B^2, C^2, D^2 and have distinguished factors $(\tau^{2n}\sigma^{2m})^{\pm 1}$ or $(\dot{\tau}^{2n}\dot{\sigma}^{2m})^{\pm 1}$, (ii) mixed terms of the type AB, AC, DB, DC, with factors of the type $(\tau^n\sigma^m)^{\pm 1}(\dot\tau^n\dot\sigma^m)^{\pm 1}$ and (iii) terms of the type AD and BC (and the constant on the RHS) in which the factors containing τ, σ, etc cancel each other.

It turns out that the above three types of coefficients separate from each other and lead to a number of relations that are independently valid on the basis of the Hirota–Miwa equations (4.7). For example the coefficient of C^2 (or $\dot\tau^{-2n}\dot\sigma^{-2m}$) is quartic in the F's, but it can be decomposed into various combinations of (4.7) as follows:

$$\begin{aligned}F_{2\bar3}\times(\text{coeff. of } C^2) \propto{}& [(4.7\text{c})_{\bar3}F_{12\bar34}-(4.7\text{d})_{\bar3}F_{12}][(a+q)F_2F_{\bar34}-(b+q)FF_{2\bar34}]\\ &-[(4.7\text{a})_{\bar3}F_{\bar34}-(4.7\text{b})_{\bar3}F)][(a-p)F_2F_{12\bar34}-(b-p)F_{12}F_{2\bar34}]\end{aligned}$$

and similarly for the coefficients of A^2, B^2, C^2. The coefficient of AB, on the other hand, is proportional to

$$\begin{aligned}&\{[-(4.7\text{b})_3F+(4.7\text{c})F_{34}][(a+p)F_2F_{123\bar4}-(b+p)F_{12}F_{23\bar4}]\\ &\quad+[(4.7\text{b})_{3\bar4}F_{12}-(4.7\text{a})F_{123\bar4}][(a-q)F_2F_{34}+(b+q)FF_{234}]\}(b-p)(b-q)\\ &\quad-\{[(4.7\text{d})_{3\bar4}F-(4.7\text{c})F_{3\bar4}][(a+p)F_2F_{1234}+(b-p)F_{12}F_{234}]\\ &\quad-[(4.7\text{d})_3F_{12}-(4.7\text{a})F_{1234}][(a-q)F_2F_{3\bar4}-(b-q)FF_{23\bar4}]\}(b+p)(b+q)\end{aligned}$$

and similar formulae can be written for the coefficients of AC, DB and DC.

This leaves the equation containing AD, BC and δ^2, which can be written as

$$AD\mathcal{R}(b)+BC\mathcal{R}(-b)+\delta^2\mathcal{S}=0 \tag{4.8}$$

where $\mathcal{R}$ and $\mathcal{S}$ are some expression in F and the parameters and $\mathcal{S}$ does not depend on b. The reflection property with respect to b is obvious when one considers the role of b in (2.3), (3.2), (4.1), (4.3).

One of the fundamental assumptions before was that although the coefficients A, B, C, D may depend on δ, the soliton part F does not. Thus equation (4.8) should hold under (4.5), whether or not $\delta=0$. If $\delta=0$ we get immediately the condition

$$(a-b)^2\mathcal{R}(b)+(a+b)^2\mathcal{R}(-b)=0 \tag{4.9}$$

and then using this we get for $\delta\neq 0$

$$\mathcal{R}(b)-16ab(a+b)^2\mathcal{S}=0. \tag{4.10}$$

If (4.10) holds, so does its $b\to -b$ reflection, and they together imply (4.9). In full detail equation (4.10) reads

$$\begin{aligned}&F_{12\bar34}F_{13\bar4}F_2F(a+p)(a-p)(a-q)(b+p)(b-p)(b+q)\\ &\quad-F_{12\bar34}F_1F_{23\bar4}F(a-p)(a+q)(a-q)(b+p)(b+q)(b-q)\\ &\quad+F_{12\bar34}F_1F_2F_{3\bar4}(a-p)(a-q)(b+p)(b+q)(p+q)(p-q)\\ &\quad+F_{123\bar4}F_{1\bar34}F_2F(a+p)(a-p)(a+q)(b+p)(b-p)(b-q)\\ &\quad-F_{123\bar4}F_1F_{2\bar34}F(a+p)(a+q)(a-q)(b-p)(b+q)(b-q)\\ &\quad+F_{123\bar4}F_1F_2F_{\bar34}(a+p)(a+q)(b-p)(b-q)(p+q)(p-q)\\ &\quad+F_{12}F_{1\bar34}F_{23\bar4}F(a-p)(a+q)(b+p)(b-q)(p+q)(p-q)\\ &\quad-F_{12}F_{1\bar34}F_2F_{3\bar4}(a-p)(a+q)(a-q)(b+p)(b+q)(b-q)\\ &\quad+F_{12}F_{13\bar4}F_{2\bar34}F(a+p)(a-q)(b-p)(b+q)(p+q)(p-q)\\ &\quad-F_{12}F_{13\bar4}F_2F_{\bar34}(a+p)(a+q)(a-q)(b-p)(b+q)(b-q)\\ &\quad+F_{12}F_1F_{2\bar34}F_{3\bar4}(a+p)(a-p)(a-q)(b+p)(b-p)(b+q)\\ &\quad+F_{12}F_1F_{23\bar4}F_{\bar34}(a+p)(a-p)(a+q)(b+p)(b-p)(b-q)\\ &\quad-4F_{12}F_1F_2F(a+b)^2(p+q)(p-q)ab=0.\end{aligned} \tag{4.11}$$

In order to prove (4.11) we need in addition to equations (4.7) another relation, namely

$$(p-a)(q+a)F_{1\bar{3}}F_{23}-(p+a)(q-a)F_{13}F_{2\bar{3}}-2a(p-q)FF_{12}=0, \tag{4.12}$$

and its $3\to 4$, $a\to b$ version. While it is straightforward to show that (4.12) is satisfied by the NSS (4.1), the implementation of this equation together with (4.7) (and their various shifts and reflections) to prove (4.11) is rather complicated and was done with the help of computer algebra (REDUCE, [23])

5. Conclusions

In this paper we have constructed the *N*-soliton solution to Q3 (2.6). This was done through an associated equation, the NQC equation (2.1), which contains extra parameters a, b. The correspondence from NQC to $(\mathrm{Q3})_0$ is 4-to-1, labeled by the different sign combinations of a, b, and this leads us to a four-term background solution (3.3), from which the 1SS (3.15) and then the NSS (4.1) is constructed. The solutions contain the new parameters a, b in an essential manner and it is intriguing that here the natural parameters seem to be p, q rather than $\mathring{p}, \mathring{q}$—a geometric interpretation to this would be interesting. An important observation is the relationship with the Hirota–Miwa equation with four variables.

A proof based on the known NSS of the NQC equation, as well as other details and properties will be given elsewhere [18].

Acknowledgments

One of the authors (JH) would like to thank JJC Nimmo and Da-Jun Zhang for discussions. JA was supported by the UK Engineering and Physical Sciences Research Council.

Appendix A. Miura transformation between $(\mathrm{Q3})_0$ and $(\mathrm{Q3})_\delta$

In section 3 we constructed a 1SS (3.15) starting with the most general 0SS (3.3) and using the CAC-cube, in which the vertical direction associated with the Bäcklund transformation. Here we again use the CAC-cube, but now the bottom layer is $(\mathrm{Q3})_\delta$, the top layer $(\mathrm{Q3})_0$, with the sides providing linear equations by which a solution of $(\mathrm{Q3})_0$ can be transformed into a solution of $(\mathrm{Q3})_\delta$. For example the well-known NSS, corresponding to $A=1, B=C=D=0$, which satisfies (3.4) but not (3.5), can be transformed into a solution of $(\mathrm{Q3})_\delta$ and therefore this transformation should generate at least a B term [18].

A.1. Derivation of Miura transformation

In order to get the $\delta=0$ version of (2.6) on the top layer of the cube we use a simple scaling argument. Let us again denote the vertical shift by a bar and introduce the scaling $\overline{u}=v/\epsilon$, with $\epsilon\to 0$, where v solves $(\mathrm{Q3})_0$. In this limit the top equation immediately becomes (2.4). For the side equation we get first

$$P(uv+\widetilde{u}\widetilde{v})-\epsilon R\left(u\widetilde{u}+\frac{1}{\epsilon^2}v\widetilde{v}\right)=(p^2-r^2)\left(u\widetilde{v}+\widetilde{u}v+\frac{\epsilon\delta^2}{4PR}\right),$$

and a similar equation for the shift in the other lattice direction associated with q. Arranging the parameter R to be of order ϵ, i.e. $R=\epsilon R_0$ with R_0 finite and nonzero as $\epsilon\to 0$, yields in dominant order in ϵ the equation

$$P(uv+\widetilde{u}\widetilde{v})-R_0v\widetilde{v}=(p^2-r^2)\left(u\widetilde{v}+\widetilde{u}v+\frac{\delta^2}{4PR_0}\right).$$

However, we must respect the parametrization of (2.5), which now reads

$$\epsilon^2 R_0^2 = (r^2 - a^2)(r^2 - b^2),$$

and this can only be of the right order in ϵ if $r^2 = a^2 + \epsilon^2 r_0^2$, or $r^2 = b^2 - \epsilon^2 r_0^2$ with r_0 finite as $\epsilon \to 0$. Hence, $R_0^2 = r_0^2(a^2 - b^2)$, but since r_0 is arbitrary, we may keep R_0 arbitrary and only remember that either $r^2 = a^2$ or $r^2 = b^2$. Thus, we are led to the following Miura transformation between $(\mathrm{Q}3)_\delta$ and $(\mathrm{Q}3)_0$:

$$P(uv + \widetilde{u}\widetilde{v}) - Rv\widetilde{v} = (p^2 - r^2)\left(\widetilde{u}v + u\widetilde{v} + \frac{\delta^2}{4PR}\right), \tag{A.1a}$$

$$Q(uv + \widehat{u}\widehat{v}) - Rv\widehat{v} = (q^2 - r^2)\left(\widehat{u}v + u\widehat{v} + \frac{\delta^2}{4QR}\right), \tag{A.1b}$$

where we have suppressed the suffix 0.

In (A.1) v is the solution of $(\mathrm{Q}3)_0$ and u a corresponding solution of $(\mathrm{Q}3)_\delta$, and we note that the resulting equations are actually *linear* in the new solution u:

$$[Pv - (p^2 - a^2)\widetilde{v}]u + [P\widetilde{v} - (p^2 - a^2)v]\widetilde{u} = Rv\widetilde{v} + \frac{\delta^2(p^2 - a^2)}{4PR}, \tag{A.2a}$$

$$[Qv - (q^2 - a^2)\widehat{v}]u + [Q\widehat{v} - (q^2 - a^2)v]\widehat{u} = Rv\widehat{v} + \frac{\delta^2(q^2 - a^2)}{4QR}, \tag{A.2b}$$

where we have chosen $r^2 = a^2$. It can be shown that the Miura transformations (A.2) commute on the two-dimensional lattice (since the lattice equations obeyed by v belong to a set of multidimensionally consistent equations), as a consequence of which we can 'integrate' the linear relations (A.2a) and (A.2b) simultaneously in both lattice directions to obtain a well-defined solution.

Other Miura-type transformations between various members of the ABS list of lattice equations, and related equations that can be obtained by degeneration have been constructed [22].

A.2. From seed of $(\mathrm{Q}3)_0$ *to the seed of* $(\mathrm{Q}3)_\delta$

We now apply the Miura transformation (A.1) starting with v given by the background solution $v_{n,m} = -\tau^n\sigma^m/(a+b)$. In that case the coefficients of the linear relation (A.2a) are given by

$$Pv - (p^2 - a^2)\widetilde{v} = \frac{P}{p-b}\tau^n\sigma^m, \qquad P\widetilde{v} - (p^2 - a^2)v = -\frac{P}{p+b}\tau^{n+1}\sigma^m, \tag{A.3}$$

and similarly for the coefficients of (A.2b) interchanging the roles of $\mathfrak{p}$ and $\mathfrak{q}$ and of n and m. Inserting these coefficients we obtain the linear equations

$$u - \frac{p-b}{p+b}\tau\widetilde{u} = \frac{R(p-b)}{P(a+b)^2}\tau^{n+1}\sigma^m + \frac{\delta^2}{4(p+b)R}\tau^{-n}\sigma^{-m}, \tag{A.4a}$$

$$u - \frac{q-b}{q+b}\sigma\widehat{u} = \frac{R(q-b)}{Q(a+b)^2}\tau^n\sigma^{m+1} + \frac{\delta^2}{4(q+b)R}\tau^{-n}\sigma^{-m}. \tag{A.4b}$$

The first relation can be integrated w.r.t. the variable n, using $\dot{\tau}^n$ defined in (3.2) as integrating factor, and we obtain

$$u_{0,m} - \dot{\tau}^n u_{n,m} = \frac{R\sigma^m}{2a(a+b)^2}\left[\left(\frac{p+a}{p-a}\right)^n - 1\right] - \frac{\delta^2\sigma^{-m}}{8bR}\left[\left(\frac{p-b}{p+b}\right)^n - 1\right].$$

What is significant here is that the coefficients on the right-hand side no longer depend on the lattice parameter p, and that p and q only occur through the discrete 'exponentials'. Obviously a similar relation can be obtained from (A.4*b*) integrating w.r.t. variable m, and the result is given by simply interchanging p ad q and the roles of n and m;

$$u_{n,0} - \dot{\sigma}^m u_{n,m} = \frac{R\tau^n}{2a(a+b)^2}\left[\left(\frac{q+a}{q-a}\right)^m - 1\right] - \frac{\delta^2\tau^{-n}}{8bR}\left[\left(\frac{q-b}{q+b}\right)^m - 1\right],$$

with $\dot{\sigma}$ defined in (2.3).

Now we can either take the first result and add a multiple of the second result at $n = 0$, or take the second result and add a multiple of the first result at $m = 0$, thus eliminating the intermediate terms $u_{0,m}$ and $u_{n,0}$ respectively. In either way we get the following result:

$$u_{n,m} = -\tau^n\sigma^m \frac{R}{2a(a+b)^2} + \tau^{-n}\sigma^{-m}\frac{\delta^2}{8bR} + \dot{\tau}^{-n}\dot{\sigma}^{-m}\left(u_{00} - \frac{\delta^2}{8bR} + \frac{R}{2a(a+b)^2}\right).$$

Thus, starting with the background solution for $(\mathrm{Q3})_0$ with $A = 1$, $B = C = D = 0$ we generated a solution of $(\mathrm{Q3})_\delta$ with $A = -R/(2a(a+b)^2)$, $B = \delta^2/(8bR)$, $C = 0$, $D \neq 0$, which indeed satisfies constraint (3.5).

This three-term solution can be suggestively written as follows:

$$u_{n,m} = A\tau^n\sigma^m + C\dot{\tau}^{-n}\dot{\sigma}^{-m} + D\tau^{-n}\sigma^{-m},$$

which contains all but one of the possible sign interchangements of the parameters a and b in the discrete exponential factors. This seems to suggest that there is one term missing, indeed, another three-term solution with A, B, D would be obtained by using a Miura transformation with $r^2 = b^2$, instead of $r^2 = a^2$, which we used in (A.2). Thus we reach ansatz (3.3).

References

[1] Ablowitz M J and Ladik F J 1976 A nonlinear difference scheme and inverse scattering *Stud. Appl. Math.* **55** 213–29
Ablowitz M J and Ladik F J 1977 On the solution of a class of nonlinear partial difference equations *Stud. Appl. Math.* **57** 1–12
[2] Hirota R 1977 Nonlinear partial difference equations I–III *J. Phys. Soc. Japan* **43** 1424–33, 2074–89
[3] Nijhoff F W, Quispel G R W and Capel H W 1983 Direct linearization of nonlinear difference–difference equations *Phys. Lett.* **97A** 125–8
[4] Quispel G R W, Nijhoff F W, Capel H W and van der Linden J 1984 Linear integral equations and nonlinear difference–difference equations *Physica* **125A** 344–80
[5] Date E, Jimbo M and Miwa T 1982 Method for generating discrete soliton equations I–II *J. Phys. Soc. Japan* **51** 4116–31
Date E, Jimbo M and Miwa T 1983 Method for generating discrete soliton equations III–V *J. Phys. Soc. Japan* **52** 388–93, 761–71
[6] Nijhoff F W and Capel H W 1995 The discrete Korteweg–de Vries equation *Acta Appl. Math.* **39** 133–58
[7] Nijhoff F W and Walker A J 2001 The discrete and continuous Painlevé VI hierarchy and the garnier systems *Glasgow Math. J.* **43A** 109–23
[8] Bobenko A I and Suris Yu B 2002 Integrable systems on quad-graphs *Int. Math. Res. Not.* **11** 573–611
[9] Adler V E, Bobenko A I and Suris Yu B 2002 Classification of integrable equations on quad-graphs, the consistency approach *Commun. Math. Phys.* **233** 513–43
[10] Adler V E, Bobenko A I and Suris Y B 2007 *Discrete Nonlinear Hyperbolic Equations. Classification of Integrable Cases* (*Preprint* 0705.1663)
[11] Adler V E 1998 Bäcklund transformation for the Krichever–Novikov equation *Int. Math. Res. Not.* **1** 1–4
[12] Krichever I M and Novikov S P 1979 Holomorphic fiberings and nonlinear equations *Sov. Math.—Dokl.* **20** 650–4
[13] Krichever I M and Novikov S P 1981 Holomorphic bundles over algebraic curves and nonlinear equations *Russ. Math. Surv.* **35** 53–79
[14] Nijhoff F W 2002 Lax pair for the Adler (lattice Krichever–Novikov) system *Phys. Lett.* A **297** 49–58

[15] Adler V E and Suris Yu B 2004 $Q4$: integrable master equation related to an elliptic curve *Int. Math. Res. Not.* **47** 2523–53

[16] Atkinson J, Hietarinta J and Nijhoff F 2007 Seed and soliton solutions of Adler's lattice equation *J. Phys. A: Math. Theor.* **40** F1–8

[17] Atkinson J and Nijhoff F W 2007 Solutions of Adler's lattice equation associated with 2-cycles of the Bäcklund transformation *J. Nonlinear Math. Phys. Proc. NEEDS 2007 Conf.* (*Preprint* 0710.2643) at press

[18] Atkinson J and Nijhoff F W in preparation

[19] Rasin O G and Hydon P 2006 Conservation laws for NQC-type difference equations *J. Phys. A: Math. Gen.* **39** 14055–66

[20] Hirota R 1981 Discrete analogue of a generalized Toda equation *J. Phys. Soc. Japan* **50** 3785–91

[21] Miwa T 1982 On Hirota's difference equations *Proc. Japan Acad.* A **58** 9–12

[22] Atkinson J 2008 Bäcklund transformations for integrable lattice equations *Preprint* 0801.1998

[23] Hearn A C 2004 REDUCE User's Manual, Version 3, 8